解析学 II　解答例

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■　曲線 $x = x(t)$，$y = y(t)$　$(a \leq t \leq b)$ はすべての t に対して $\{x'(t)\}^2 + \{y'(t)\}^2 > 0$ をみたすものとする．関数

$$s = h(t) = \int_0^t \sqrt{\{x'(z)\}^2 + \{y'(z)\}^2}\, dz, \qquad a \leq t \leq b$$

の逆関数を $t = h^{-1}(s)$ とするとき，関数 $p(s) = x(h^{-1}(s))$，$q(s) = y(h^{-1}(s))$ を s に関して 2 回微分せよ．

(解)　$R(s) = \sqrt{\{x'(h^{-1}(s))\}^2 + \{y'(h^{-1}(s))\}^2}$ とおくと，逆関数の微分法により

$$\frac{dh^{-1}}{ds}(s) = \frac{1}{\sqrt{\{x'(h^{-1}(s))\}^2 + \{y'(h^{-1}(s))\}^2}} = \frac{1}{R(s)}$$

が得られ，合成関数の微分法により

$$\frac{dp}{ds}(s) = x'(h^{-1}(s))\,\frac{dh^{-1}}{ds}(s) = \frac{x'(h^{-1}(s))}{R(s)}, \qquad \frac{dq}{ds}(s) = y'(h^{-1}(s))\,\frac{dh^{-1}}{ds}(s) = \frac{y'(h^{-1}(s))}{R(s)}$$

がみたされる．

$$\begin{aligned}
\frac{dR}{ds}(s) =& \frac{1}{2\sqrt{\{x'(h^{-1}(s))\}^2 + \{y'(h^{-1}(s))\}^2}} \\
& \qquad \times \left\{2\,x'(h^{-1}(s))\,x''(h^{-1}(s))\,\frac{dh^{-1}}{ds}(s) + 2\,y'(h^{-1}(s))\,y''(h^{-1}(s))\,\frac{dh^{-1}}{ds}(s)\right\} \\
=& \frac{x'(h^{-1}(s))\,x''(h^{-1}(s)) + y'(h^{-1}(s))\,y''(h^{-1}(s))}{\{R(s)\}^2}
\end{aligned}$$

より

$$\begin{aligned}
\frac{d^2p}{ds^2}(s) =& \frac{\left\{x''(h^{-1}(s))\,\dfrac{dh^{-1}}{ds}(s)\right\} \cdot R(s) - x'(h^{-1}(s)) \cdot \dfrac{dR}{ds}(s)}{\{R(s)\}^2} \\
=& \frac{y'(h^{-1}(s))}{R(s)} \cdot \frac{x''(h^{-1}(s))\,y'(h^{-1}(s)) - x'(h^{-1}(s))\,y''(h^{-1}(s))}{\{R(s)\}^3}, \\
\frac{d^2q}{ds^2}(s) =& \frac{\left\{y''(h^{-1}(s))\,\dfrac{dh^{-1}}{ds}(s)\right\} \cdot R(s) - y'(h^{-1}(s)) \cdot \dfrac{dR}{ds}(s)}{\{R(s)\}^2} \\
=& -\frac{x'(h^{-1}(s))}{R(s)} \cdot \frac{x''(h^{-1}(s))\,y'(h^{-1}(s)) - x'(h^{-1}(s))\,y''(h^{-1}(s))}{\{R(s)\}^3}
\end{aligned}$$

となる．■