解析学 II　解答例

2017.04.24

■　3 次元ベクトル $\mathbf{p} = (p_j)$，$\mathbf{q} = (q_j)$ の外積 $\mathbf{p} \times \mathbf{q}$ を求めよ．

(解)　基底 $\mathbf{e}_1 = (1,0,0)^T$，$\mathbf{e}_2 = (0,1,0)^T$，$\mathbf{e}_3 = (0,0,1)^T$ を用いて

$$\mathbf{p} = p_1\,\mathbf{e}_1 + p_2\,\mathbf{e}_2 + p_3\,\mathbf{e}_3, \qquad \mathbf{q} = q_1\,\mathbf{e}_1 + q_2\,\mathbf{e}_2 + q_3\,\mathbf{e}_3$$

と表せ，

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \qquad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \qquad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2, \qquad \mathbf{e}_j \times \mathbf{e}_j = \mathbf{0} \quad (j = 1,2,3)$$

であるから，

$$\begin{aligned}
\mathbf{p} \times \mathbf{q} =& (p_1\,\mathbf{e}_1 + p_2\,\mathbf{e}_2 + p_3\,\mathbf{e}_3) \times (q_1\,\mathbf{e}_1 + q_2\,\mathbf{e}_2 + q_3\,\mathbf{e}_3) \\
=& p_1\,q_1\,\mathbf{e}_1 \times \mathbf{e}_1 + p_2\,q_1\,\mathbf{e}_2 \times \mathbf{e}_1 + p_3\,q_1\,\mathbf{e}_3 \times \mathbf{e}_1 \\
& + p_1\,q_2\,\mathbf{e}_1 \times \mathbf{e}_2 + p_2\,q_2\,\mathbf{e}_2 \times \mathbf{e}_2 + p_3\,q_2\,\mathbf{e}_3 \times \mathbf{e}_2 \\
& + p_1\,q_3\,\mathbf{e}_1 \times \mathbf{e}_3 + p_2\,q_3\,\mathbf{e}_2 \times \mathbf{e}_3 + p_3\,q_3\,\mathbf{e}_3 \times \mathbf{e}_3 \\
=& - p_2\,q_1\,\mathbf{e}_3 + p_3\,q_1\,\mathbf{e}_2 + p_1\,q_2\,\mathbf{e}_3 - p_3\,q_2\,\mathbf{e}_1 - p_1\,q_3\,\mathbf{e}_2 + p_2\,q_3\,\mathbf{e}_1 \\
=& (p_2\,q_3 - p_3\,q_2)\,\mathbf{e}_1 - (p_1\,q_3 - p_3\,q_1)\,\mathbf{e}_2 + (p_1\,q_2 - p_2\,q_1)\,\mathbf{e}_3 = \det \begin{pmatrix} p_1 & q_1 & \mathbf{e}_1 \\ p_2 & q_2 & \mathbf{e}_2 \\ p_3 & q_3 & \mathbf{e}_3 \end{pmatrix}
\end{aligned}$$

が得られる．■