解析学 II　解答例

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■　漸化式

$$\begin{pmatrix} p_0 \\ q_0 \\ r_0 \\ s_0 \\ t_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1/4 & 0 & 0 & 0 \\ 4/4 & 0 & 2/4 & 0 & 0 \\ 0 & 3/4 & 0 & 3/4 & 0 \\ 0 & 0 & 2/4 & 0 & 4/4 \\ 0 & 0 & 0 & 1/4 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \\ r_n \\ s_n \\ t_n \end{pmatrix}, \qquad n = 0, 1, 2, \cdots$$

で定められる点列 $\{(p_n, q_n, r_n, s_n, t_n)\}$ の一般項を求めよ．

（解）関係式

$$p_{n+1} = \frac{q_n}{4}, \qquad r_{n+1} = \frac{3(q_n + s_n)}{4}, \qquad t_{n+1} = \frac{s_n}{4}, \qquad n = 0, 1, 2, \cdots$$

より q_n，s_n を求めれば良い．$q_0 = 0$，$s_0 = 0$，$q_1 = 1$，$s_1 = 0$ より，

$$q_n + s_n = p_{n-1} + r_{n-1} + t_{n-1} = q_{n-2} + s_{n-2} = \cdots = \begin{cases} q_0 + s_0 = 0 & (n : \text{偶数}), \\ q_1 + s_1 = 1 & (n : \text{奇数}), \end{cases}$$

$$q_n - s_n = \quad p_{n-1} - t_{n-1} = \frac{q_{n-2} - s_{n-2}}{2^2} = \cdots \quad = \begin{cases} \dfrac{q_0 - s_0}{2^n} = 0 & (n : \text{偶数}), \\ \dfrac{q_1 - s_1}{2^{n-1}} = 2^{1-n} & (n : \text{奇数}) \end{cases}$$

が得られ，

$$q_n = \begin{cases} 0 & (n : \text{偶数}), \\ \dfrac{2^{n-1} + 1}{2^n} & (n : \text{奇数}), \end{cases} \qquad s_n = \begin{cases} 0 & (n : \text{偶数}), \\ \dfrac{2^{n-1} - 1}{2^n} & (n : \text{奇数}) \end{cases}$$

となる．したがって，$n \in \mathbb{N}$ に対して

$$p_n = \frac{q_{n-1}}{4} = \begin{cases} 0 & (n : \text{奇数}), \\ \dfrac{2^{n-2} + 1}{2^{n+1}} & (n : \text{偶数}), \end{cases}$$

$$r_n = \frac{3(q_{n-1} + s_{n-1})}{4} = \begin{cases} 0 & (n : \text{奇数}), \\ \dfrac{3}{4} & (n : \text{偶数}), \end{cases}$$

$$t_n = \frac{s_{n-1}}{4} = \begin{cases} 0 & (n : \text{奇数}), \\ \dfrac{2^{n-2} - 1}{2^{n+1}} & (n : \text{偶数}) \end{cases}$$

と表せる．■